Available online at www.sciencedirect.com
SCIENCE DIRECT®
Stochastic Processes and their Applications 116 (2006) 643–661
stochastic processes and their applications
www.elsevier.com/locate/spa

Existence of densities for jumping stochastic differential equations

Nicolas Fournier, Jean-Sébastien Giet*

Institut Elie Cartan, Campus Scientifique, BP 239, 54506 Vandoeuvre-lès-Nancy Cedex, France

Received 10 November 2004; received in revised form 7 September 2005; accepted 14 November 2005
Available online 15 December 2005

Abstract

We consider a jumping Markov process $\{X_t^x\}_{t\geq 0}$. We study the absolute continuity of the law of X_t^x for $t > 0$. We first consider, as Bichteler and Jacod [K. Bichteler, J. Jacod, Calcul de Malliavin pour les diffusions avec sauts, existence d'une densité dans le cas unidimensionel, in: Séminaire de Probabilités XVII, in: L.N.M., vol. 986, Springer, 1983, pp. 132–157] did, the case where the rate of jumping is constant. We state some results in the spirit of those of [K. Bichteler, J. Jacod, Calcul de Malliavin pour les diffusions avec sauts, existence d'une densité dans le cas unidimensionel, in: Séminaire de Probabilités XVII, in: L.N.M., vol. 986, Springer, 1983, pp. 132–157], with rather weaker assumptions and simpler proofs, not relying on the use of stochastic calculus of variations. We next extend our method to the case where the rate of jumping depends on the spatial variable, and this last result seems to be new.

MSC: 60H10; 60J75

Keywords: Stochastic differential equations; Jump processes; Absolute continuity

1. Introduction

Consider a d-dimensional Markov process with jumps $\{X_t^x\}_{t\geq 0}$, starting from $x \in \mathbb{R}^d$, with generator $\mathcal{L}$, defined for $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ sufficiently smooth and $y \in \mathbb{R}^d$, by

$$\mathcal{L}\phi(y) = b(y).\nabla\phi(y) + \int_{\mathbb{R}^n} \gamma(y)\varphi(z)\,[\phi(y + h(y, z)) - \phi(y)]\,\mathrm{d}z, \tag{1.1}$$

with possibly an additional diffusion term, and the integral part written in a (more general) *compensated* form. Here $n \in \mathbb{N}$ is fixed, and the functions $\gamma : \mathbb{R}^d \mapsto \mathbb{R}$ and $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$

* Corresponding author. Tel.: +33 (0)3 83 68 45 45; fax: +33 (0)3 83 68 45 34.
E-mail addresses: fournier@iecn.u-nancy.fr (N. Fournier), giet@iecn.u-nancy.fr (J.-S. Giet).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.11.002

are non-negative. We aim to investigate the absolute continuity of the law of X_t^x with respect to the Lebesgue measure on $\mathbb{R}^d$, for $t > 0$. We will sometimes allow the presence of a Brownian part, but we will actually not use the regularizing effect of the Brownian motion.

Assume for a moment that $d = n = 1$. Roughly speaking, the law of X_t^x is expected to have a density as soon as $t > 0$, if for all $y \in \mathbb{R}$, $\gamma(y) \int_{\mathbb{R}} \varphi(z)\mathrm{d}z = \infty$ and if $h(y, z)$ is not too constant in z (for example $h(y, .)$ of class C^1 with a non-zero derivative almost everywhere). Indeed, in such a case, X^x has infinitely many jumps immediately after $t = 0$. Furthermore, the jumps are of the shape $X_t^x = X_{t-}^x + h(X_{t-}^x, Z)$, with Z a random variable independent of X_{t-}^x, with law $\varphi(z)\mathrm{d}z$. This produces absolute continuity for X_t^x, if h is sufficiently non-constant in z.

This simple idea is not so easy to handle rigorously, since X^x has infinitely many jumps, and since $\varphi(z)\mathrm{d}z$ is not a probability measure (because $\int_{\mathbb{R}^n} \varphi(z)\mathrm{d}z = \infty$). To our knowledge, all the known results are based on the use of *stochastic calculus of variations*, i.e. on a sort of *differential calculus* with respect to the stochastic variable ω. The first results in this direction were obtained by Bismut [3]. Important results are due to Bichteler and Jacod [2], and then Bichteler, Gravereaux and Jacod [1]. We refer the reader to Graham and Méléard [7] and Fournier and Giet [6] for applications to physical integro-differential equations such as the Boltzmann and the coagulation–fragmentation equations. See also [11] and [4] for alternative methods in the much more complicated case where the intensity measure of N is singular.

All the previously cited works concern the case where the *rate of jumping* $\gamma(x)$ is constant. The case where γ is non-constant is much more delicate. The main reason for this is that in such a case, the map $x \mapsto X_t^x$ cannot be regular (or even continuous). Indeed, if $\gamma(x) < \gamma(y)$, and if $\int_{\mathbb{R}^n} \varphi(z)\mathrm{d}z = \infty$, then it is clear that for all small $t > 0$, X^y jumps infinitely more often than X^x before t. The only available result with γ not constant seems to be that of [5], for which the assumptions are very restrictive: monotonicity (in x) is assumed for h and γ.

First, we would like to give some results in the spirit of Bichteler and Jacod [2], with simpler proofs. We will in particular not use the stochastic calculus of variations. We thus consider in Section 2 the case where γ is constant. We state and prove a result under a strong non-degeneracy assumption on h. The proof is elementary, and our result follows the lines of Theorem 2.5 in [2], but our assumptions are rather weaker. We will next extend our result to the case where γ is not constant in Section 3. This last result seems to be new, and improves those of [5].

Our method allows us to improve slightly the results of [2] concerning the existence of a density when γ is constant, and to obtain a result when γ depends on the variable position. Let us however recall that for when γ is constant, the methods of [2] were extended in [1] to study the existence of a density under weaker non-degeneracy assumptions, and to establish the smoothness of the density. Our method does not seem to promise such extensions.

Throughout the whole paper, we denote the collection of Lebesgue-null Borelian subsets of $\mathbb{R}^d$ by

$$\mathcal{A} = \left\{ A \in \mathcal{B}(\mathbb{R}^d);\ \int_A \mathrm{d}x = 0 \right\}. \tag{1.2}$$

2. The case of a constant rate of jumping

Consider the following d-dimensional S.D.E., for some $d \in \mathbb{N}$, starting from $x \in \mathbb{R}^d$:

$$X_t^x = x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} h(X_{s-}^x, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \sigma(X_s^x)\mathrm{d}B_s, \tag{2.1}$$

where

Assumption (I)**.** $N(\mathrm{d}s, \mathrm{d}z)$ is a Poisson measure on $[0, \infty) \times \mathbb{R}^n$, for some $n \in \mathbb{N}$, with intensity measure $\nu(\mathrm{d}s, \mathrm{d}z) = \mathrm{d}s\varphi(z)\mathrm{d}z$. The function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}_+$ is supposed to be measurable. We denote by $\tilde{N} = N - \nu$ the associated *compensated* Poisson measure. The $\mathbb{R}^m$-valued (for some $m \in \mathbb{N}$) Brownian motion $\{B_t\}_{t\geq 0}$ is supposed to be independent of N.

In this case, the generator of the Markov process X^x is given, for any $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ sufficiently smooth, for any $y \in \mathbb{R}^d$, by

$$\mathcal{L}\phi(y) = b(y).\nabla\phi(y) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{i,j}(x)\partial_i\partial_j\phi(x) + \int_{\mathbb{R}^n}[\phi(y + h(y, z)) - \phi(y) - h(y, z).\nabla\phi(y)]\,\varphi(z)\mathrm{d}z. \tag{2.2}$$

We assume the following hypothesis, $\mathcal{M}_{d\times m}$ standing for the set of $d \times m$ matrices with real entries.

Assumption $(H1)$**.** The functions $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto \mathcal{M}_{d\times m}$ are of class C^2 and have at most linear growth. The function $h : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}^d$ is measurable. For each $z \in \mathbb{R}^n$, $x \mapsto h(x, z)$ is of class C^2 on $\mathbb{R}^d$. There exists $\eta \in L^2(\mathbb{R}^n, \varphi(z)\mathrm{d}z)$ and a continuous function $\zeta : \mathbb{R}^d \mapsto \mathbb{R}$ such that for all $x \in \mathbb{R}^d, z \in \mathbb{R}^n$, $|h(x, z)| \leq (1 + |x|)\eta(z)$, while $|h'_x(x, z)| + |h''_{xx}(x, z)| \leq \zeta(x)\eta(z)$.

Then $\mathcal{L}\phi$ is well defined for all $\phi \in C_b^2(\mathbb{R}^d)$ and it is well known that the following result holds.

Proposition 2.1. *Assume* (I) *and* $(H1)$*. Consider the natural filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *associated with the Poisson measure* N *and the Brownian motion* B*. Then, for any* $x \in \mathbb{R}^d$*, there exists a unique càdlàg* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process* $\{X_t^x\}_{t\geq 0}$ *solution to* (2.1) *such that for all* $x \in \mathbb{R}^d$*, for all* $T \in [0, \infty)$*,*

$$E\left[\sup_{s\in[0,T]}|X_s^x|^2\right] < \infty. \tag{2.3}$$

The process $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ *furthermore satisfies the strong Markov property.*

See [8] for the case of globally Lipschitz coefficients. A standard localization procedure allows us to obtain Proposition 2.1.

We now study the absolute continuity of the law of X_t^x for $t > 0$. We first give some assumptions, statements, and examples. The proof is handled in a second part.

2.1. Statements

We first introduce some assumptions. Here I_d stands for the unit $d \times d$ matrix, while x_0 is a given point of $\mathbb{R}^d$.

Assumption $(H2)$**.** For all $x \in \mathbb{R}^d$, for all $z \in \mathbb{R}^n$, $\det(I_d + h'_x(x, z)) \neq 0$.

Assumption $(H3)(x_0)$**.** There exists $\epsilon > 0$ such that for all $x \in B(x_0, \epsilon)$, there exists a subset $O(x) \subset \mathbb{R}^n$ such that (recall (1.2)),

$$\int_{O(x)}\varphi(z)\mathrm{d}z = \infty, \qquad \text{and for all } A \in \mathcal{A}, \quad \int_{O(x)}\mathbf{1}_{\{h(x,z)\in A\}}\varphi(z)\mathrm{d}z = 0, \tag{2.4}$$

and such that the map $(x, z) \mapsto \mathbf{1}_{\{z\in O(x)\}}$ is measurable on $B(x_0, \epsilon) \times \mathbb{R}^n$.

The main results of this section are the following.

Theorem 2.2. *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (I), (H_1), (H_2) and $(H_3)(x_0)$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

In the case where $(H3)(x)$ holds for all $x \in \mathbb{R}^d$, we can omit assumption $(H2)$.

Corollary 2.3. *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (I), (H_1) and that $(H_3)(x)$ holds for all $x \in \mathbb{R}^d$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

We do not state a result concerning the regularizing effect of the Brownian part of (2.1), since it seems reasonable that standard techniques of Malliavin calculus (see, e.g., [10]) may allow one to prove that under $(H1)$, $(H2)$ and if $\sigma\sigma^*(x_0)$ is invertible, then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.

These results subsequently relax the assumptions of [2] concerning the regularity (in z) and boundedness conditions on h. However, a mixed non-degeneracy condition connecting h and σ was assumed in [2]: it seems hard to obtain such a result without using the Malliavin calculus at least for the Brownian part.

Let us comment on our hypotheses. The second condition in (2.4) means that the image measure of $\mathbf{1}_{\{z\in O(x)\}}\varphi(z)\mathrm{d}z$ (where $\mathrm{d}z$ stands for the Lebesgue measure on $\mathbb{R}^n$), by the map $z \mapsto h(x, z)$, has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, for each $x \in B(x_0, \epsilon)$. Let us now state a typical case of application.

Proposition 2.4. *Assume that $n = d$, that $x_0 \in \mathbb{R}^d$, and that there exist $\epsilon > 0$ and an open subset O of $\mathbb{R}^n$ such that $(x, z) \mapsto h(x, z)$ is of class C^1 on $B(x_0, \epsilon) \times O$. If for all $x \in B(x_0, \epsilon)$, $\int_O \mathbf{1}_{\{\det h'_z(x,z)\neq 0\}}\varphi(z)\mathrm{d}z = \infty$, then $(H3)(x_0)$ holds.*

Indeed, it suffices to note that, since $n = d$, $h'_z(x, z)$ is a $d\times d$ matrix for each $x \in \mathbb{R}^d$, for each $z \in O$. Choosing $O(x) = \{z \in O,\ \det h'_z(x, z) \neq 0\}$ for $x \in B(x_0, \epsilon)$ allows us to conclude, noting that, due to the local inverse theorem, the map $z \mapsto h(x, z)$ is a local C^1-diffeomorphism on $O(x)$.

Assumptions $(H1)$ and $(H3)(x_0)$ are quite natural. Note that $(H2)$ is not just a technical condition, as this example shows.

Example 2.5. Assume that $n = d = 1$, that $\varphi \equiv 1$, that b, σ satisfy $(H1)$ with $b(0) = \sigma(0) = 0$, and that $h(x, z) = -x\mathbf{1}_{\{|z|\leq 1\}} + (x/|z|)\mathbf{1}_{\{|z|>1\}}$. Then (I) and $(H1)$ are satisfied, while $(H3)(x)$ holds for all $x \neq 0$, but $(H2)$ fails. One can prove that in such a case, $P[X_t^{x_0} = 0] > 0$ for all $t > 0$, and thus the law of $X_t^{x_0}$ is not absolutely continuous. Indeed, it is clear that, if $T_1 = \inf\{t \geq 0;\ \int_0^t\int_{\mathbb{R}} \mathbf{1}_{\{|z|\leq 1\}}N(\mathrm{d}s, \mathrm{d}z) \geq 1\}$, then T_1 has an exponential distribution with parameter 2, and $X_{T_1} = X_{T_1-} + (-X_{T_1-}) = 0$. Since furthermore $b(0) = \sigma(0) = 0$ and $h(0, .) = 0$, a uniqueness argument and the strong Markov property show that $X_{T_1+t} = 0$ a.s. for all $t \geq 0$. Hence $P[X_t^{x_0} = 0] \geq P[T_1 < t] = 1 - \mathrm{e}^{-2t} > 0$ for all $t > 0$.

2.2. Proof

We now turn to the proof of Theorem 2.2. We first proceed to a localization procedure.

Lemma 2.6. *To prove Theorem 2.2 and Corollary 2.3, we may assume the additional condition* $(H4)$ *below.*

Assumption $(H4)$**.** The functions $b, b', b'', \sigma, \sigma', \sigma''$ are bounded. There exists $\tilde{\eta} \in L^2(\mathbb{R}^n, \varphi(z)\mathrm{d}z)$ such that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^n$, $|h(x,z)| + |h'_x(x,z)| + |h''_{xx}(x,z)| \le \tilde{\eta}(z)$.

Proof. We study the case of Theorem 2.2. Let $x_0 \in \mathbb{R}^d$ be fixed. Assume that Theorem 2.2 holds under (I), (H_1), (H_2), $(H_3)(x_0)$, $(H4)$, and consider some functions b, σ, h satisfying only (I), (H_1), (H_2), $(H_3)(x_0)$. For each $l \ge 1$, consider some functions b_l, σ_l, h_l satisfying (I), (H_1), (H_2), $(H_3)(x_0)$, $(H4)$ and such that for all $|x| \le l$, for all $z \in \mathbb{R}^n$, $b_l(x) = b(x)$, $\sigma_l(x) = \sigma(x)$, and $h_l(x,z) = h(x,z)$. Denote by $\{X_t^{x_0,l}\}_{t\ge 0}$ the solution to (2.1) with h, σ, b replaced by h_l, σ_l, b_l. Then, by assumption, the law of $X_t^{x_0,l}$ has a density if $t > 0$. Next, denote by $\tau_l = \inf\{t \ge 0,\ |X_t^{x_0}| \ge l\}$. It is clear from (2.3) that τ_l increases a.s. to infinity as l tends to infinity. Furthermore a uniqueness argument yields that a.s. $X_t^{x_0} = X_t^{x_0,l}$ for all $t \le \tau_l$. Hence, for any $A \in \mathcal{A}$, and any $t > 0$, by the Lebesgue Theorem,

$$\begin{aligned} P[X_t^{x_0} \in A] &= \lim_{l\to\infty} P[X_t^{x_0} \in A, t < \tau_l] \\ &= \lim_{l\to\infty} P[X_t^{x_0,l} \in A, t < \tau_l] \le \lim_{l\to\infty} P[X_t^{x_0,l} \in A] = 0, \end{aligned} \tag{2.5}$$

since the law of $X_t^{x_0,l}$ has a density for each $l \ge 1$. This implies that the law of $X_t^{x_0}$ has a density. $\square$

We now gather some known results about the *flow* $\{X_t^x\}_{t\ge 0, x\in\mathbb{R}^d}$.

Lemma 2.7. *Assume* (I), $(H1)$, $(H4)$. *Consider the flow of solutions* $\{X_t^x\}_{t\ge 0, x\in\mathbb{R}^d}$ *to* (2.1). *Then a.s., the map* $x \mapsto X_t^x$ *is of class* C^1 *on* $\mathbb{R}^d$ *for each* $t \ge 0$. *If furthermore* $(H2)$ *holds, then a.s., for all* $t \ge 0$, *and all* $x \in \mathbb{R}^d$, $\det \frac{\partial}{\partial x} X_t^x \ne 0$.

Proof. It is well known (see [12] Theorems 39 and 40 Section 7 for very similar results) that under (I), $(H1)$, $(H4)$, the map $x \mapsto X_t^x$ is a.s. of class C^1 on $\mathbb{R}^d$ for each $t \ge 0$, and that one may differentiate (2.1) with respect to x:

$$\begin{aligned} \frac{\partial}{\partial x} X_t^x &= I_d + \int_0^t b'(X_s^x) \frac{\partial}{\partial x} X_s^x \mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} h'_x(X_{s-}^x, z) \frac{\partial}{\partial x} X_{s-}^x \tilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &\quad + \int_0^t \sigma'(X_s^x) \frac{\partial}{\partial x} X_s^x \mathrm{d}B_s. \end{aligned} \tag{2.6}$$

Then, following the ideas of Jacod ([9], Theorem 1 and Corollary page 443), we deduce an explicit expression for $V_t^x = \det \frac{\partial}{\partial x} X_t^x$ in terms of Doléans–Dade exponentials (a continuity argument shows that this explicit expression holds a.s. simultaneously for all $x \in \mathbb{R}^d$). We thus obtain, still using the results of [9] simultaneously for all $x \in \mathbb{R}^d$, that a.s., $\det \frac{\partial}{\partial x} X_t^x \ne 0$ for all $x \in \mathbb{R}^d$ and all $t < T^x$, where

$$T^x = \inf\left\{ t \ge 0;\ \int_0^t \int_{\mathbb{R}^n} \mathbf{1}_{\{\det(I_d + h'_x(X_{s-}^x, z)) = 0\}} N(\mathrm{d}s, \mathrm{d}z) \ge 1 \right\}. \tag{2.7}$$

But $(H2)$ ensures that a.s., $T^x = \infty$ for all $x \in \mathbb{R}^d$. $\square$

We may now prove Theorem 2.2.

Proof of Theorem 2.2. Due to Lemma 2.6, we suppose the additional condition $(H4)$. We consider $x_0 \in \mathbb{R}^d$ and $t > 0$ fixed.

Step 1. Due to $(H3)(x_0)$, we may build, for each $x \in B(x_0, \epsilon)$, an increasing sequence $\{O_p(x)\}_{p\geq 1}$ of subsets of $\mathbb{R}^n$ such that

$$\cup_{p\geq 1} O_p(x) = O(x) \qquad \text{and } \forall\, p \geq 1, \qquad \int_{O_p(x)} \varphi(z)\mathrm{d}z = p, \tag{2.8}$$

in such a way that for each $p \geq 1$, the map $(x, z) \mapsto \mathbf{1}_{\{z\in O_p(x)\}}$ is measurable on $B(x_0, \epsilon) \times \mathbb{R}^n$.

We also consider the stopping time

$$\tau = \inf\{s \geq 0;\ |X_s^{x_0} - x_0| \geq \epsilon\} > 0 \text{ a.s.} \tag{2.9}$$

The positivity of τ comes from the fact that X^{x_0} is a.s. right-continuous and starts from x_0.

We finally consider the stopping time, for $p \geq 1$,

$$S_p = \inf\left\{s \geq 0;\ \int_0^s \int_{\mathbb{R}^n} \mathbf{1}_{\left\{z\in O_p\left(X^{x_0}_{(u\wedge\tau)-}\right)\right\}} N(\mathrm{d}u, \mathrm{d}z) \geq 1\right\}, \tag{2.10}$$

and the associated *mark* $Z_p \in \mathbb{R}^n$, uniquely defined by $N(\{S_p\} \times \{Z_p\}) = 1$.

Due to (2.8), and to the fact that $X^{x_0}_{(u\wedge\tau)-}$ always belongs to $B(x_0, \epsilon)$, one may prove that

(i) $p \mapsto S_p$ is a.s. non-increasing,
(ii) $\lim_{p\to\infty} S_p = 0$ a.s.,
(iii) conditionally to $\mathcal{F}_{S_p-}$, the law of Z_p is given by $\frac{1}{p}\varphi(z)\mathbf{1}_{\left\{z\in O_p\left(X^{x_0}_{(S_p\wedge\tau)-}\right)\right\}}\mathrm{d}z$, where

$$\mathcal{F}_{S_p-} = \sigma\left(B \cap \{S_p > s\};\ s \geq 0,\ B \in \mathcal{F}_s\right). \tag{2.11}$$

Indeed, (i) is obvious by construction, since $p \mapsto O_p(x)$ is increasing for each $x \in \mathbb{R}^d$. Next, an easy computation shows that the compensator of the (random) point measure $N^p(\mathrm{d}s, \mathrm{d}z) = \mathbf{1}_{\{z\in O_p(X^{x_0}_{(s\wedge\tau)-})\}} N(\mathrm{d}s, \mathrm{d}z)$ is given by $p\mathrm{d}s \times p^{-1}\mathbf{1}_{\{z\in O_p(X^{x_0}_{(s\wedge\tau)-})\}}\varphi(z)\mathrm{d}z$. Since for each $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^n} p^{-1}\mathbf{1}_{\{z\in O_p(x)\}}\varphi(z)\mathrm{d}z = 1$, we deduce that the *rate* of jumping of N^p is constant and equal to p, so that S_p, which is the first instant of the jump, has an exponential distribution with parameter p. This and (i) ensure (ii). We also obtain (iii) as a consequence of the shape of the compensator of N^p.

Step 2. We now prove that conditionally to $\sigma(S_p)$, the law of $X^{x_0}_{S_p}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, on the set $\Omega^0_p = \{\tau \geq S_p\}$. Since S_p is $\mathcal{F}_{S_p-}$-measurable, it suffices to prove that for any $A \in \mathcal{A}$, a.s., $P[\Omega^0_p, X^{x_0}_{S_p} \in A \mid \mathcal{F}_{S_p-}] = 0$. But, using the notation of Step 1, a.s., $X^{x_0}_{S_p} = X^{x_0}_{S_p-} + h[X^{x_0}_{S_p-}, Z_p]$ on Ω^0_p. Furthermore, we know that on Ω^0_p, $X^{x_0}_{S_p-} \in B(x_0, \epsilon)$ a.s. Thus, using Step 1 (see (iii)), since $\{\tau \geq S_p\}$ and $X^{x_0}_{S_p-}$ are $\mathcal{F}_{S_p-}$-measurable,

$$\begin{aligned} P[\Omega^0_p, X^{x_0}_{S_p} \in A \mid \mathcal{F}_{S_p-}] &= \mathbf{1}_{\Omega^p_0} P\left[X^{x_0}_{S_p-} + h[X^{x_0}_{S_p-}, Z_p] \in A \mid \mathcal{F}_{S_p-}\right] \\ &= \mathbf{1}_{\left\{\tau\geq S_p, X^{x_0}_{S_p-}\in B(x_0,\epsilon)\right\}} \int_{\mathbb{R}^n} \mathbf{1}_{\left\{h\left[X^{x_0}_{S_p-}, z\right]\in A - X^{x_0}_{S_p-}\right\}} \\ &\quad \times \frac{1}{p}\varphi(z)\mathbf{1}_{\left\{z\in O_p\left(X^{x_0}_{S_p-}\right)\right\}}\mathrm{d}z = 0 \end{aligned} \tag{2.12}$$

due to $(H3)(x_0)$ (use (2.4) with $x = X^{x_0}_{S_p-}$), since for any $y \in \mathbb{R}^d$, $A - y = \{x - y, x \in A\}$ belongs to $\mathcal{A}$.

Step 3. We may now deduce that for any $p \geq 1$, the law of $X^{x_0}_t$ has a density on the set $\Omega^1_p = \{S_p \leq \tau \wedge t\}$. We deduce from Step 2 that on $\Omega^1_p \subset \Omega^0_p$ the law of $(S_p, X^{x_0}_{S_p})$ is of the shape $\nu_p(\mathrm{d}s) f_p(s, x)\mathrm{d}x$, where ν_p is the law of S_p while $f_p(s, .)$ is the density of $X^{x_0}_{S_p}$ conditionally to $S_p = s$.

Hence, for any $A \in \mathcal{A}$, using the strong Markov property, we obtain, conditioning with respect to $\mathcal{F}_{S_p}$,

$$P[\Omega^1_p, X^{x_0}_t \in A] = E\left[\mathbf{1}_{\Omega^1_p} E\left\{\int_0^t \nu_p(\mathrm{d}s) \int_{\mathbb{R}^d} f_p(s, x)\mathrm{d}x \mathbf{1}_{\{X^x_{t-s} \in A\}}\right\}\right]. \tag{2.13}$$

It thus suffices to show that a.s., for any $s < t$ fixed,

$$\int_{\mathbb{R}^d} f_p(s, x)\mathrm{d}x \mathbf{1}_{\{X^x_{t-s} \in A\}} = 0. \tag{2.14}$$

Of course, it suffices to check that a.s., for $s < t$ fixed,

$$\int_{\mathbb{R}^d} \mathrm{d}x \mathbf{1}_{\{X^x_{t-s} \in A\}} = 0. \tag{2.15}$$

But this is immediate from Lemma 2.7, using that the Jacobian of the map $x \mapsto X^x_{t-s}$ (a.s.) never vanishes and that A is Lebesgue-null: one may find, due to the local inverse theorem, a countable family of open subsets R_i of $\mathbb{R}^d$, on which $x \mapsto X^x_{t-s}$ is a C^1 diffeomorphism, and such that $\mathbb{R}^d = \cup_{i=1}^{\infty} R_i$. The conclusion follows, performing the substitution $x \mapsto y = X^x_{t-s}$ on each R_i. This allows us to conclude that $P[\Omega^p_1, X^{x_0}_t \in A] = 0$.

Step 4. The conclusion readily follows: due to Step 1 (see (2.9) and (ii)), $\mathbf{1}_{\Omega^1_p}$ goes a.s. to 1 as p tends to infinity. We thus infer from the Lebesgue Theorem that for any $A \in \mathcal{A}$,

$$P[X^{x_0}_t \in A] = \lim_{p\to\infty} P[\Omega^1_p, X^{x_0}_t \in A] = 0, \tag{2.16}$$

thanks to Step 3. Thus the law of $X^{x_0}_t$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$.

□

We finally show how to relax assumption $(H2)$ when $(H3)(x)$ holds everywhere.

Proof of Corollary 2.3. Due to Lemma 2.6, we may suppose the additional assumption $(H4)$. We consider $\delta_0 > 0$ such that for any $M \in \mathcal{M}_{d\times d}$ satisfying $|M| \leq \delta_0$, $\det(I_d + M) \geq 1/2$. We then split $\mathbb{R}^n$ into $O_F \cup O_I$, with

$$O_F = \left\{z \in \mathbb{R}^n,\ \tilde{\eta}(z) \geq \delta_0\right\}, \quad O_I = \left\{z \in \mathbb{R}^n,\ \tilde{\eta}(z) < \delta_0\right\}. \tag{2.17}$$

Since $\tilde{\eta} \in L^2(\mathbb{R}^n, \varphi(z)\mathrm{d}z)$, we deduce that $\lambda_F := \int_{O_F} \varphi(z)\mathrm{d}z \leq \delta_0^{-2} \int_{\mathbb{R}^n} \tilde{\eta}^2(z)\varphi(z)\mathrm{d}z < \infty$. We next consider the solution $\{Y^x_t\}_{t\geq 0}$ to the S.D.E.

$$\begin{aligned} Y^x_t &= x + \int_0^t b(Y^x_s)\mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} h(Y^x_{s-}, z)\mathbf{1}_{\{z\in O_I\}}\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &\quad - \int_0^t \int_{O_F} h(Y^x_{s-}, z)\varphi(z)\mathrm{d}z\mathrm{d}s + \int_0^t \sigma(Y^x_s)\mathrm{d}B_s. \end{aligned} \tag{2.18}$$

Clearly, this S.D.E. satisfies (I), $(H1)$, (H_2), and $(H3)(x)$ for all x, so that due to Theorem 2.2, the law of Y_t^x has a density for each $t > 0$, and each $x \in \mathbb{R}^d$. The solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1) may now be realized in the following way (see [8] for details): consider a standard Poisson process with intensity λ_F and with instants of the jump $0 = T_0 < T_1 < T_2 < \cdots$, a family of i.i.d. $\mathbb{R}^n$-valued random variables $(Z_i)_{i\geq 1}$ with law $\lambda_F^{-1}\varphi(z)\mathbf{1}_{\{z\in O_F\}}\mathrm{d}z$, and a family of i.i.d. solutions $(\{Y_t^{i,x}\}_{x\in\mathbb{R}^d,t\geq 0})_{i\geq 1}$ to (2.18), all these random objects being independent. Set

$$X_0^{x_0} = x_0, \quad \forall\, i \geq 0, \quad X_{T_{i+1}}^{x_0} = Y_{T_{i+1}-T_i}^{i,X_{T_i}^{x_0}} + h(Y_{T_{i+1}-T_i}^{i,X_{T_i}^{x_0}}, Z_i) \quad \text{and}$$

$$\forall\, t \geq 0,\ X_t^{x_0} = \sum_{i\geq 0} Y_{t-T_i}^{i,X_{T_i}^{x_0}} \mathbf{1}_{\{t\in[T_i,T_{i+1})\}}. \tag{2.19}$$

Then $\{X_t^{x_0}\}_{t\geq 0}$ is a solution (in law) to (2.1). To conclude, notice that for any $t > 0$, one has $t \notin \cup_i\{T_i\}$ a.s., so that for any $A \in \mathcal{A}$,

$$\begin{aligned} P[X_t^{x_0} \in A] &= \sum_{i\geq 0} P\left[Y_{t-T_i}^{i,X_{T_i}^{x_0}} \in A,\ t \in (T_i, T_{i+1})\right] \\ &\leq \sum_{i\geq 0} P\left[Y_{t-T_i}^{i,X_{T_i}} \in A, t > T_i\right] = 0. \end{aligned} \tag{2.20}$$

The last equality comes from the facts that for each i, $\{Y_s^{i,x}\}_{s\geq 0,x\in\mathbb{R}^d}$ is independent of (T_i, X_{T_i}), and that the law of $Y_t^{i,x}$ has a density for each $t > 0$, for each $x \in \mathbb{R}^d$. $\square$

Let us conclude this section with a remark. One may obtain, with such a method, some results in the spirit of [1]: the existence of a density under some non-degeneracy assumptions less stringent than $(H3)$. The main idea is to use a finite number of jumps (instead of one), and the method becomes quite complicated. The result that we can obtain is slightly better than that of [1] from the point of view of the regularity of h. However, we cannot obtain some mixed non-degeneracy conditions on the Brownian motion and on the Poisson measure, which was possible in [1], and which makes the result of [1] much more general.

3. The case of a non-constant rate of jumping

Consider now the following d-dimensional S.D.E., for some $d \in \mathbb{N}$, starting from $x \in \mathbb{R}^d$:

$$X_t^x = x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} \int_0^\infty h(X_{s-}^x, z)\mathbf{1}_{\{u\leq\gamma(X_{s-}^x)\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \tag{3.1}$$

where

Assumption (J)**.** $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson measure on $[0,\infty)\times\mathbb{R}^n\times[0,\infty)$, for some $n \in \mathbb{N}$, with intensity measure $\nu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}s\varphi(z)\mathrm{d}z\mathrm{d}u$. The function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}_+$ is supposed to be measurable.

In this case, the generator of the Markov process X^x is given, for any $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ sufficiently smooth and $y \in \mathbb{R}^d$, by

$$\mathcal{L}\phi(y) = b(y).\nabla\phi(y) + \int_{\mathbb{R}^n} \gamma(y)\left[\phi(y + h(y, z)) - \phi(y)\right]\varphi(z)\mathrm{d}z. \tag{3.2}$$

It might be possible to add a Brownian term and consider a compensated Poisson measure. However, the present situation simplifies the computations. We assume the following hypothesis.

Assumption $(A1)$**.** The function $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ is of class C^1, and has at most linear growth. The function $\gamma : \mathbb{R}^d \mapsto \mathbb{R}_+$ is of class C^1. The function $h : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}^d$ is measurable. For each $z \in \mathbb{R}^n$, $x \mapsto h(x,z)$ is of class C^1 on $\mathbb{R}^d$. There exists $\eta \in L^1(\mathbb{R}^n, \varphi(z)\mathrm{d}z)$ and a continuous function $\zeta : \mathbb{R}^d \mapsto \mathbb{R}$ such that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^n$, $\gamma(x)|h(x,z)| \le (1+|x|)\eta(z)$, while $|h'_x(x,z)| \le \zeta(x)\eta(z)$.

Then $\mathcal{L}\phi$ is well defined for all $\phi \in C_b^1(\mathbb{R}^d)$, and it is well known that the following result holds.

Proposition 3.1. *Assume (J) and $(A1)$. Consider the natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ associated with the Poisson measure N. Then, for any $x \in \mathbb{R}^d$, there exists a unique càdlàg $\{\mathcal{F}_t\}_{t\ge 0}$-adapted process $\{X_t^x\}_{t\ge 0}$ solution to (3.1) such that for all $x \in \mathbb{R}^d$, for all $T \in [0,\infty)$,*

$$E\left[\sup_{s\in[0,T]} |X_s^x|\right] < \infty. \tag{3.3}$$

The process $\{X_t^x\}_{t\ge 0, x\in\mathbb{R}^d}$ furthermore satisfies the strong Markov property.

We refer the reader to [5] (Section 2) for the proof of a very similar result. We divide the section into three parts: we start with the statements and proofs, and we end with an example of application.

3.1. Statements

To obtain some absolute continuity results, we will assume the following conditions. Here x_0 is fixed in $\mathbb{R}^d$.

Assumption $(A2)$**.** There exists $c_0 \in (0,1)$ such that for all $x \in \mathbb{R}^d$, for all $z \in \mathbb{R}^n$, $\det(I_d + h'_x(x,z)) \ge c_0$. For each $z \in \mathbb{R}^n$, the map $x \mapsto x + h(x,z)$ is a C^1-diffeomorphism.

Remark that if $d = 1$, the condition $1 + h'_x(x,z) \ge c_0 > 0$ ensures that $(A2)$ holds.

Assumption $(A3)(x_0)$**.** The function γ never vanishes. There exists $\epsilon > 0$ such that for all $x \in B(x_0,\epsilon)$, there exists a subset $O(x) \subset \mathbb{R}^n$ such that (recall (1.2))

$$\int_{O(x)} \varphi(z)\mathrm{d}z = \infty, \qquad \text{and for all } A \in \mathcal{A}, \quad \int_{O(x)} \mathbf{1}_{\{h(x,z)\in A\}}\varphi(z)\mathrm{d}z = 0, \tag{3.4}$$

and such that the map $(x,z) \mapsto \mathbf{1}_{\{z\in O(x)\}}$ is measurable on $B(x_0,\epsilon) \times \mathbb{R}^n$.

The main results of this section are the following.

Theorem 3.2. *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (J), $(A1)$, $(A2)$ and $(A3)(x_0)$. Consider the unique solution $\{X_t^{x_0}\}_{t\ge 0}$ to (3.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

As previously, an immediate consequence is the following.

Corollary 3.3. *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (J), $(A1)$ and that $(A3)(x)$ holds for all $x \in \mathbb{R}^d$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (3.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

These results consequently improve those of [5], where many restrictive conditions were assumed, such as the monotonicity of $x \mapsto \gamma(x)$ and $x \mapsto h(x,z)$, and the positivity of $h(x,z)$.

Exactly as in Section 2 (see Proposition 2.4), we have a general example of application, using the local inverse theorem.

Proposition 3.4. *Assume (J) and $(A1)$, and let $x_0 \in \mathbb{R}^d$. Suppose that $n = d$, and that γ never vanishes. Assume that there exists $\epsilon > 0$ and an open subset $O \subset \mathbb{R}^d$ such that h is of class C^1 on $B(x_0,\epsilon) \times O$. If for all x in $B(x_0,\epsilon)$, $\int_O \mathbf{1}_{\{\det h'_z(x,z)\neq 0\}}\varphi(z)\mathrm{d}z = \infty$, then $(A3)(x_0)$ holds.*

3.2. Proof

First of all, we proceed to a localization procedure.

Lemma 3.5. *To prove Theorem 3.2 and Corollary 3.3, we may assume the additional condition $(A4)$ below.*

Assumption $(A4)$**.** The functions b, b', γ and γ' are bounded. There exists $\tilde\eta \in L^1(\mathbb{R}^n, \varphi(z)\mathrm{d}z)$ such that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^n$, $|h(x,z)| + |h'_x(x,z)| \leq \tilde\eta(z)$. There exists $\gamma_0 > 0$ such that for all $x \in \mathbb{R}^d$, $\gamma(x) \geq \gamma_0$.

We omit the proof of this lemma, since it is the same as that of Lemma 2.6 (see also [5] Section 2). We will need the following lemma.

Lemma 3.6. (i) *There exists $\beta_0 > 0$ such that for any C^1 function $\delta : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying $\|\delta'\|_\infty \leq \beta_0$, the map $x \mapsto x + \delta(x)$ is a C^1-diffeomorphism, and for all $x \in \mathbb{R}^d$, $\det(I_d + \delta'(x)) \geq 1/2$.*
(ii) *There exist some constants $\theta_1 \in (0,1)$ and $\theta_2 > 0$ such that for all $M \in \mathcal{M}_{d\times d}$ with $|M| \leq \theta_1$, $\det(I_d + M) \geq 1 - \theta_2|M| \geq 1/2$.*

Proof. (i) Set $\zeta(x) = x + \delta(x)$. First of all, it is clear, by continuity of the determinant, that if β_0 is small enough, $\det\zeta'(x) = \det[I_d + \delta'(x)] \geq 1/2$ for all $x \in \mathbb{R}^d$. Thus, it classically suffices to show that, if β_0 is small enough, ζ is injective. Consider thus x, y such that $\zeta(x) = \zeta(y)$. Then $|x - y| = |\delta(x) - \delta(y)| \leq \|\delta'\|_\infty|x - y| \leq \beta_0|x - y|$, which implies that $x = y$ if $\beta_0 < 1$.

(ii) We use the norm $|M| = \sup_{i,j}|M_{i,j}|$. A rough computation shows that, if $|M| \leq 1$,

$$\det(I_d + M) \geq (1 - |M|)^d - d!\sum_{i=0}^{d-1}(1 - |M|)^i|M|^{d-i} \geq 1 - d|M| - d!d|M|. \tag{3.5}$$

The result follows, setting $\theta_2 = d(1 + d!)$ and $\theta_1 = 1/2d(1 + d!)$. □

Next, we note that the proof of Corollary 3.3 is the same as that of Corollary 2.3, using of course Theorem 3.2 instead of Theorem 2.2, and using β_0 defined in Lemma 3.6 rather than δ_0. We thus omit the proof of Corollary 2.3.

The main novelty of this section consists in the following proposition, which allows us to overcome the irregularity of the map $x \mapsto X_t^x$.

Proposition 3.7. *Assume* (J), $(A1)$, $(A2)$ *and* $(A4)$, *and denote by* $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ *the unique solution to* (3.1). *Consider a probability density function* f_0 *on* $\mathbb{R}^d$. *Then for all* $t \geq 0$, *for all* $A \in \mathcal{A}$,

$$\int_{\mathbb{R}^d} f_0(x) P[X_t^x \in A] \mathrm{d}x = 0. \tag{3.6}$$

In other words, if X_0 is a random variable (independent of N) with law $f_0(x)\mathrm{d}x$, then $X_t^{X_0}$ has a density for each $t \geq 0$. To prove this, we first consider the case where f_0 satisfies some additional conditions.

Lemma 3.8. *Assume* (J), $(A1)$, $(A2)$ *and* $(A4)$, *and denote by* $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ *the unique solution to* (3.1). *Consider a* d*-dimensional random variable* X_0, *independent of* N, *satisfying* $E[|X_0|] < \infty$. *Assume that the law of* X_0 *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$, *and that its density* f_0 *satisfies*

$$\int_{\mathbb{R}^d} f_0^2(x)\mathrm{d}x < \infty. \tag{3.7}$$

Then for all $t \geq 0$, *the law of* $X_t^{X_0}$ *has a density* $f(t,x)$, *and furthermore, for any* $T \in [0,\infty)$,

$$\sup_{[0,T]} \int_{\mathbb{R}^d} f^2(t,x)\mathrm{d}x < \infty. \tag{3.8}$$

Proof. We split the proof into several steps. We first introduce an approximating process X_t^l in Step 1. We next show some non-uniform L^∞ estimates for the density of X_t^l in Step 2, which allow us to prove rigorously some uniform (in l) L^2 estimates in Step 3. We go to the limit in Step 4.

Step 1. We consider a sequence $\{f_l^0\}_{l\geq 1}$ of bounded and continuous probability density functions, converging to f_0 in $L^2(\mathbb{R}^d)$. We build a sequence $\{X_0^l\}_{l\geq 1}$ of random variables (independent of N), such that for each l, the law of X_0^l is given by $f_l^0(x)\mathrm{d}x$. Since $E[|X_0|] < \infty$, we may handle this construction in such a way that $\lim_l E[|X_0 - X_0^l|] = 0$. We also consider an increasing sequence K_l of subsets of $\mathbb{R}^n$ such that $\cup_l K_l = \operatorname{supp} \tilde{\eta}$ (recall $(A4)$), and such that for each l, $\Lambda_l = \int_{K_l} \varphi(z)\mathrm{d}z < \infty$ (choose for example $K_l = \{z \in \mathbb{R}^n, \tilde{\eta}(z) \geq 1/l\}$). We finally denote, for each $l \in \mathbb{N}$, by $\{X_t^l\}_{t\geq 0}$ a $\mathbb{R}^d$-valued Markov process starting from X_0^l and with generator $\mathcal{L}^l$, defined for any bounded measurable function $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ and any $x \in \mathbb{R}^d$, by

$$\begin{aligned}\mathcal{L}^l\phi(x) &= l\gamma(x)\left[\phi(x + b(x)/l\gamma(x)) - \phi(x)\right] \\ &\quad + \gamma(x)\int_{K_l} \varphi(z)\mathrm{d}z \left[\phi(x + h[x,z]) - \phi(x)\right].\end{aligned} \tag{3.9}$$

We now show that for each $t \geq 0$, X_t^l converges to X_t in law as l tends to infinity. To this end, we build $\{X_t^l\}_{t\geq 0}$ with the help of N, and of another independent Poisson measure $M^l(\mathrm{d}s, \mathrm{d}u)$ on $[0,\infty) \times [0,\infty)$ with intensity measure $l\mathrm{d}s\mathrm{d}u$:

$$X_t^l = X_0^l + \int_0^t \int_0^\infty \frac{b(X_{s-}^l)}{l\gamma(X_{s-}^l)} \mathbf{1}_{\{u \leq \gamma(X_{s-}^l)\}} M^l(\mathrm{d}s, \mathrm{d}u)$$

$$+\int_0^t \int_{K_l} \int_0^\infty h(X^x_{s-}, z)\mathbf{1}_{\{u\le\gamma(X_{s-})\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \tag{3.10}$$

Noting that

$$Y^l_t = \int_0^t \int_0^\infty \frac{b(X^l_{s-})}{l\gamma(X^l_{s-})}\mathbf{1}_{\{u\le\gamma(X^l_{s-})\}} M^l(\mathrm{d}s, \mathrm{d}u) - \int_0^t b(X^l_s)\mathrm{d}s \tag{3.11}$$

is a martingale with bracket

$$\left\langle Y^l\right\rangle_t = \frac{1}{l}\int_0^t \frac{b^2(X^l_s)}{\gamma(X^l_s)}\mathrm{d}s \le \|b/\gamma\|^2_\infty \frac{t}{l} \to 0, \tag{3.12}$$

and using $(A1)$ and $(A4)$ repeatedly, one may then show that for any $T \ge 0$,

$$\lim_{l\to\infty} E\left[\sup_{[0,T]} |X^l_t - X^{X_0}_t|\right] = 0. \tag{3.13}$$

Step 2. Consider now $l_0 > \|(b/\gamma)'\|_\infty/\beta_0$, where β_0 was defined in Lemma 3.6(i). This is possible due to $(A4)$. We aim to prove that for any $l \ge l_0$, and any $t \ge 0$, X^l_t has a bounded density $f_l(t, x)$, and that for any $T > 0$,

$$\sup_{[0,T]} \sup_{x\in\mathbb{R}^d} f_l(t, x) < \infty. \tag{3.14}$$

We thus consider $l \ge l_0$ to be fixed. We also denote, for any $a \in (0, \infty)$, by $\mathcal{C}_a = \{A \in \mathcal{B}(\mathbb{R}^d);\ \int_A \mathrm{d}x \le a\}$. A direct computation, using (3.10), the fact that γ is bounded, and neglecting all the non-positive terms, yields that there exists a constant C (depending on l) such that for any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} P[X^l_t \in A] &= P[X^l_0 \in A] + l\int_0^t E\left[\gamma(X^l_s)\left\{\mathbf{1}_{\{X^l_s + b(X^l_s)/l\gamma(X^l_s)\in A\}} - \mathbf{1}_{\{X^l_s\in A\}}\right\}\right]\mathrm{d}s \\ &\quad + \int_0^t \int_{K_l} E\left[\gamma(X^l_s)\left\{\mathbf{1}_{\{X^l_s + h(X^l_s,z)\in A\}} - \mathbf{1}_{\{X^l_s\in A\}}\right\}\right]\varphi(z)\mathrm{d}z\mathrm{d}s \\ &\le P[X^l_0 \in A] + C\int_0^t P[X^l_s + b(X^l_s)/l\gamma(X^l_s) \in A]\mathrm{d}s \\ &\quad + C\int_0^t \sup_{z\in K_l} P[X^l_s + h(X^l_s, z) \in A]\mathrm{d}s. \end{aligned} \tag{3.15}$$

For $A \in \mathcal{B}(\mathbb{R}^d)$, set $\tau(A) = \{x \in \mathbb{R}^d,\ x + b(x)/l\gamma(x) \in A\}$, and $\tau_z(A) = \{x \in \mathbb{R}^d,\ x + h(x, z) \in A\}$. Then, using $(A2)$, we deduce that for any $z \in \mathbb{R}^n$, for any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\int_{\tau_z(A)} \mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{1}_{\{x+h(x,z)\in A\}}\mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{1}_{\{y\in A\}} \frac{\mathrm{d}y}{|\det(I_d + h'_x(x, z))|} \le \frac{1}{c_0}\int_A \mathrm{d}x. \tag{3.16}$$

In the same way, using that $l \ge l_0$ and Lemma 3.6(i), we get

$$\int_{\tau(A)} \mathrm{d}x \le 2\int_A \mathrm{d}x. \tag{3.17}$$

Gathering (3.15)–(3.17), we obtain, setting $n_0 = [2 \vee 1/c_0] + 1$, that for some constant C, for any $a \in (0, \infty)$,

$$\sup_{A \in \mathcal{C}_a} P[X_t^l \in A] \leq \sup_{A \in \mathcal{C}_a} P[X_0^l \in A] + C \int_0^t \sup_{A \in \mathcal{C}_{n_0 a}} P[X_s^l \in A] \mathrm{d}s$$
$$\leq a \| f_l^0 \|_\infty + n_0 C \int_0^t \sup_{A \in \mathcal{C}_a} P[X_s^l \in A] \mathrm{d}s. \tag{3.18}$$

To obtain the last term, we have used that any $A \in \mathcal{C}_{n_0 a}$ may be written as a union of n_0 elements of $\mathcal{C}_a$. We finally obtain, using the Gronwall Lemma, that for any T, there exists $C_{T,l}$ such that for all $a \in (0, \infty)$,

$$\sup_{[0,T]} \sup_{A \in \mathcal{C}_a} P[X_t^l \in A] \leq C_{T,l} \times a. \tag{3.19}$$

This ensures (3.14).

Step 3. We now show, and it is the heart of the proof, that for any $T \geq 0$, there exists a constant C_T, not depending on $l \geq l_0$, such that

$$\sup_{[0,T]} \int_{\mathbb{R}^d} f_l^2(t, x) \mathrm{d}x \leq C_T. \tag{3.20}$$

We will rather work with the weight function $\gamma(x)$, which seems artificial: we are however not able to conclude, working directly with $\int f_l^2 \mathrm{d}x$. We consider $l \geq l_0$ to be fixed, and we set for simplicity $\gamma f_l(t, x) = \gamma(x) f_l(t, x)$.

Step 3.1. We first show rigorously, and this is a purely technical issue, that for all $t \geq 0$,

$$\int_{\mathbb{R}^d} \gamma(x) f_l^2(t, x) \mathrm{d}x = \int_{\mathbb{R}^d} \gamma(x) (f_l^0)^2(x) \mathrm{d}x$$
$$+ 2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} f_l(t, x) \mathcal{L}^l \{\gamma f_l(t, .)\}(x) \mathrm{d}x. \tag{3.21}$$

Using Lemma 3.6(i) and that $l \geq l_0$, we may define the inverse function τ_1^l of $x \mapsto x + b(x)/l\gamma(x)$, and denote by J_1^l the associated Jacobian function. Using $(A2)$, we may also define, for all fixed $z \in K_l$, the inverse function $\tau_2^l(., z)$ of $x \mapsto x + h(x, z)$, and denote by $J_2^l(., z)$ the associated Jacobian function. Let $B_b(\mathbb{R}^d)$ denote the set of Borelian bounded functions on $\mathbb{R}^d$. We now define, for $g \in B_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mathrm{d}x)$ and $y \in \mathbb{R}^d$,

$$\mathcal{L}^{l*} g(y) = -\gamma(y) \left(l + \int_{K_l} \varphi(z) \mathrm{d}z \right) g(y) + l \gamma(\tau_1^l(y)) \left(J_1^l(\tau_1^l(y)) \right)^{-1} g(\tau_1^l(y))$$
$$+ \int_{K_l} \gamma(\tau_2^l(y, z)) \left(J_2^l(\tau_2^l(y, z), z) \right)^{-1} g(\tau_2^l(y, z)) \varphi(z) \mathrm{d}z. \tag{3.22}$$

One easily checks that for $\phi \in B_b(\mathbb{R}^d)$ and $g \in B_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mathrm{d}x)$, $\mathcal{L}^l \phi \in B_b(\mathbb{R}^d)$ while $\mathcal{L}^{l*} g \in B_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mathrm{d}x)$, and

$$\int_{\mathbb{R}^d} g(x) \mathcal{L}^l \phi(x) \mathrm{d}x = \int_{\mathbb{R}^d} \phi(y) \mathcal{L}^{l*} g(y) \mathrm{d}y. \tag{3.23}$$

We may now prove (3.21). We know from the classical theory that for all $\phi \in B_b(\mathbb{R}^d)$, for all $t \geq 0$,

$$\int_{\mathbb{R}^d} \phi(x) f_l(t,x)\mathrm{d}x = \int_{\mathbb{R}^d} \phi(x) f_l^0(x)\mathrm{d}x + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} f_l(s,x)\mathcal{L}^l \phi(x)\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \phi(x) f_l^0(x)\mathrm{d}x + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \phi(x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x. \quad (3.24)$$

Using now that for $t \geq 0$ fixed, $\gamma f^l(t,.) \in B_b(\mathbb{R}^d)$ (due to Step 2), and then that $\gamma f_l^0 \in B_b(\mathbb{R}^d)$ while for all $s \geq 0$ fixed, $\gamma \mathcal{L}^{l*} f^l(s,.) \in B_b(\mathbb{R}^d)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)\mathrm{d}x &= \int_{\mathbb{R}^d} \gamma(x) f_l^0(x) f_l(t,x)\mathrm{d}x \\ &\quad + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \gamma(x) f_l(t,x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \gamma(x) f_l^0(x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x \\ &\quad + \int_0^t \mathrm{d}s \Bigg(\int_{\mathbb{R}^d} \gamma(x) f_l^0(x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x \\ &\quad + \int_0^t \mathrm{d}u \int_{\mathbb{R}^d} \gamma(x)\mathcal{L}^{l*} f_l(u,.)(x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x \Bigg).\end{aligned} \quad (3.25)$$

Using now a symmetry argument in the last term, and then that $\gamma \mathcal{L}^{l*}(f_l(s,.)) \in B_b(\mathbb{R}^d)$ for each $s \geq 0$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)\mathrm{d}x &= \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x \\ &\quad + 2\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \gamma(x) f_l^0(x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x \qquad (3.26) \\ &\quad + 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \int_{\mathbb{R}^d} \gamma(x)\mathcal{L}^{l*} f_l(u,.)(x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x + 2\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \gamma(x) f_l(s,x)\mathcal{L}^{l*} f_l(s,.)(x)\mathrm{d}x, \qquad (3.27)\end{aligned}$$

from which (3.21) follows immediately.

Step 3.2. We may now prove (3.20). Using (3.21), we first get

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)\mathrm{d}x &= \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x \\ &\quad + 2\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} l f_l(s,x)\gamma(x)[\gamma f_l(s,x+b(x)/l\gamma(x)) - \gamma f_l(s,x)]\mathrm{d}x \\ &\quad + 2\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} f_l(s,x)\gamma(x) \int_{K_l} \varphi(z)\,[\gamma f_l(s,x+h(x,z)) - \gamma f_l(s,x)]\,\mathrm{d}z\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x + 2\int_0^t A_l(s)\mathrm{d}s + 2\int_0^t B_l(s)\mathrm{d}s,\end{aligned} \quad (3.28)$$

the last equality standing for a definition. First, using the Cauchy–Schwartz inequality, we obtain, setting $\|g\|_2^2 = \int_{\mathbb{R}^d} g^2(x)\mathrm{d}x$,

$$A_l(t) \le l\left[\|\gamma f_l(t,.)\|_2 \|\gamma f_l(t,.+b(.)/l\gamma(.))\|_2 - \|\gamma f_l(t,.)\|_2^2\right]. \tag{3.29}$$

But the substitution $x \mapsto y = x + b(x)/l\gamma(x)$, which is valid for $l \ge l_0$ due to Lemma 3.6, leads to the conclusion that

$$\begin{aligned}\|\gamma f_l(t,.+b(.)/l\gamma(.))\|_2^2 &= \int_{\mathbb{R}^d} (\gamma f_l)^2(t,y)\frac{1}{\det(I_d + (b/\gamma)'(\tau_1^l(y))/l)}\mathrm{d}y \\ &\le \frac{\|\gamma f_l(t,.)\|_2^2}{\inf\limits_{x\in\mathbb{R}^d} \det(I_d + (b/\gamma)'(x)/l)}.\end{aligned} \tag{3.30}$$

Using that $(b/\gamma)'$ is bounded due to $(A4)$, that $l \ge l_0$, and Lemma 3.6(i)–(ii), we deduce that for some constant C, not depending on l,

$$\begin{aligned}&\|\gamma f_l(t,.+b(.)/l\gamma(.))\|_2^2 \le \|\gamma f_l(t,.)\|_2^2 \\ &\quad\times\left(\frac{1}{1-\theta_2\|(b/\gamma)'\|_\infty/l}\mathbf{1}_{\{\|(b/\gamma)'\|_\infty/l<\theta_1\}} + 2\mathbf{1}_{\{\|(b/\gamma)'\|_\infty/l\ge\theta_1\}}\right) \\ &\le \|\gamma f_l(t,.)\|_2^2 \times (1 + C/l).\end{aligned} \tag{3.31}$$

We finally obtain, for some constants C_1, C_2, for any $l \ge l_0$,

$$A_l(t) \le \|\gamma f_l(t,.)\|_2^2 \times l\left[\sqrt{1+C_1/l} - 1\right] \le C_2\|\gamma f_l(t,.)\|_2^2. \tag{3.32}$$

Next, using the Fubini Theorem and then the Cauchy–Schwarz inequality, we get

$$\begin{aligned}B_l(t) &= \int_{K_l}\varphi(z)\mathrm{d}z\int_{\mathbb{R}^d}\left[\gamma f_l(t,x)\gamma f_l(t,x+h(x,z)) - (\gamma f_l)^2(t,x)\right]\mathrm{d}x \\ &\le \int_{K_l}\varphi(z)\mathrm{d}z\left[\|\gamma f_l(t,.)\|_2\|\gamma f_l(t,.+h(.,z))\|_2 - \|\gamma f_l(t,.)\|_2^2\right].\end{aligned} \tag{3.33}$$

But the substitution $x \mapsto y = x + h(x,z)$, valid due to $(A2)$, shows that

$$\|\gamma f_l(t,.+h(.,z))\|_2^2 = \int_{\mathbb{R}^d}(\gamma f_l)^2(t,y)\frac{1}{\det(I_d + h'_x(x,z))}\mathrm{d}y \le \alpha(z)\|\gamma f_l(t,.)\|_2^2, \tag{3.34}$$

where $\alpha(z) = \sup_{x\in\mathbb{R}^d}[1/\det(I_d + h'_x(x,z))]$ is well defined due to $(A2)$. We thus obtain, with the notation $r_+ = \max(x,0)$, that

$$\begin{aligned}B_l(t) &\le \|\gamma f_l(t,.)\|_2^2\int_{K_l}\varphi(z)\mathrm{d}z\left(\sqrt{\alpha(z)}-1\right)_+ \\ &\le \|\gamma f_l(t,.)\|_2^2\int_{\mathbb{R}^n}\varphi(z)\mathrm{d}z\left(\sqrt{\alpha(z)}-1\right)_+ = C\|\gamma f_l(t,.)\|_2^2.\end{aligned} \tag{3.35}$$

The constant C is finite here due to $(A2)$ and $(A4)$: one may check, using Lemma 3.6(ii) that

$$\left(\sqrt{\alpha(z)}-1\right)_+ \le \left((1-\theta_2\tilde{\eta}(z))^{-1/2}-1\right)\mathbf{1}_{\{\tilde{\eta}(z)<\theta_1\}} + \left(\frac{1}{\sqrt{c_0}}-1\right)\mathbf{1}_{\{\tilde{\eta}(z)\ge\theta_1\}}$$

$$\leq \theta_2 \tilde{\eta}(z) \mathbf{1}_{\{\tilde{\eta}(z)<\theta_1\}} + \frac{1}{\sqrt{c_0}} \mathbf{1}_{\{\tilde{\eta}(z)\geq\theta_1\}}$$
$$\leq \left(\frac{1}{\theta_1\sqrt{c_0}} \vee \theta_2 \right) \tilde{\eta}(z) \in L^1(\mathbb{R}^n, \varphi(z)\mathrm{d}z). \tag{3.36}$$

Using finally (3.28), (3.32) and (3.35), and that γ is bounded, we obtain, for some constant C not depending on $l \geq l_0$,

$$\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)\mathrm{d}x \leq \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x + C\int_0^t \|\gamma f_l(s,.)\|_2^2 \mathrm{d}s$$
$$\leq \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x + C\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \gamma(x) f_l^2(s,x)\mathrm{d}x. \tag{3.37}$$

Since γ is bounded, we deduce that $\sup_l \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)\mathrm{d}x < \infty$. Furthermore, we deduce from (3.14) that for all $T \geq 0$, for each $l \geq l_0$, $\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)\mathrm{d}x$ is bounded on $[0, T]$. We thus may conclude, using the Gronwall Lemma and the fact that γ is bounded below, that for any T,

$$\sup_{l\geq l_0} \sup_{[0,T]} \int_{\mathbb{R}^d} f_l^2(t,x)\mathrm{d}x \leq \gamma_0^{-1} \sup_{l\geq l_0} \sup_{[0,T]} \int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)\mathrm{d}x < \infty. \tag{3.38}$$

Step 4. We now fix $t \geq 0$. The finite balls of $L^2(\mathbb{R}^d)$ being weakly compact, using (3.38) allows us to find a subsequence $f_{k_l}(t,.)$, going weakly to a function $f(t,.) \in L^2(\mathbb{R}^d)$. On the other hand, we know that X_t^l converges in law to $X_t^{X_0}$. Hence the law of $X_t^{X_0}$ is given by $f(t,x)\mathrm{d}x$, and (3.38) allows us to conclude that (3.8) holds. □

Proposition 3.7 follows easily from Lemma 3.8.

Proof of Proposition 3.7. For each $n \in \mathbb{N}$, consider the probability density function f_0^n on $\mathbb{R}^d$ defined by $f_0^n(x) = c_n[f_0(x) \wedge n]\mathbf{1}_{\{|x|\leq n\}}$. Here c_n is a normalization constant. Consider a random variable X_0^n, independent of N, with law $f_0^n(x)\mathrm{d}x$. Then X_0^n satisfies the assumptions of Lemma 3.8, for each $n \in \mathbb{N}$. Thus $X_t^{X_0^n}$ has a density for each $t \geq 0$, which implies that for all $n \in \mathbb{N}$, for all $t \geq 0$, for all $A \in \mathcal{A}$,

$$\int_{\mathbb{R}^d} [f_0(x) \wedge n]\mathbf{1}_{\{|x|\leq n\}} P[X_t^x \in A]\mathrm{d}x$$
$$= c_n^{-1} \int_{\mathbb{R}^d} f_0^n(x) P[X_t^x \in A]\mathrm{d}x = c_n^{-1} P[X_t^{X_0^n} \in A] = 0. \tag{3.39}$$

The Lebesgue Theorem allows us to conclude that (3.6) holds, since $[f_0(x) \wedge n]\mathbf{1}_{\{|x|\leq n\}}$ increases pointwise to $f_0(x)$ as n tends to infinity. □

We are finally able to conclude.

Proof of Theorem 3.2. Due to Lemma 3.5, we assume the additional condition $(A4)$, and we in particular denote by $\gamma_0 > 0$ a lower bound of γ. We consider $x_0 \in \mathbb{R}^d$ and $t > 0$ to be fixed. The proof follows closely the lines of that of Theorem 2.2, so we will only sketch it.

Step 1. Due to $(A3)(x_0)$, we may build, for each $x \in B(x_0, \epsilon)$, an increasing sequence $\{O_p(x)\}_{p\geq 1}$ of subsets of $\mathbb{R}^n$ satisfying (2.8), in such a way that for each $p \geq 1$, the map

$(x, z) \mapsto \mathbf{1}_{\{z \in O_p(x)\}}$ is measurable on $B(x_0, \epsilon) \times \mathbb{R}^n$. We also consider the a.s. positive stopping time $\tau > 0$ defined by (2.9). We finally consider the stopping time, for $p \geq 1$,

$$S_p = \inf \left\{ s \geq 0;\ \int_0^s \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_{\left\{z \in O_p\left(X^{x_0}_{(r\wedge\tau)-}\right)\right\}} \mathbf{1}_{\{u \leq \gamma_0\}} N(\mathrm{d}r, \mathrm{d}z, \mathrm{d}u) \geq 1 \right\}, \tag{3.40}$$

and the associated *mark* $Z_p \in \mathbb{R}^n$, uniquely defined by $N(\{S_p\} \times \{Z_p\} \times [0, \infty)) = 1$. Due to (2.8), and to the fact that $X^{x_0}_{(u\wedge\tau)-}$ always belongs to $B(x_0, \epsilon)$, one may prove that (see the proof of Theorem 2.2 Step 1 for details)

(i) $p \mapsto S_p$ is a.s. non-increasing,
(ii) $\lim_{p\to\infty} S_p = 0$ a.s.,
(iii) conditionally to $\mathcal{F}_{S_p-}$, the law of Z_p is given by $\frac{1}{p}\varphi(z)\mathbf{1}_{\left\{z \in O_p\left(X^{x_0}_{(S_p\wedge\tau)-}\right)\right\}}\mathrm{d}z$.

Step 2. We now claim that conditionally to $\sigma(S_p)$, the law of $X^{x_0}_{S_p}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, on the set $\Omega^0_p = \{\tau \geq S_p\}$. It indeed suffices to follow line by line Step 2 of the proof of Theorem 2.2.

Step 3. We may now deduce that for any $p \geq 1$, the law of $X^{x_0}_t$ has a density on the set $\Omega^1_p = \{S_p \leq \tau \wedge t\}$. We deduce from Step 2 that on $\Omega^1_p \subset \Omega^0_p$ the law of $(S_p, X^{x_0}_{S_p})$ is of the shape $\nu_p(\mathrm{d}s) f_p(s, x)\mathrm{d}x$. Hence, for any $A \in \mathcal{A}$, using the strong Markov property, we obtain, conditioning with respect to $\mathcal{F}_{S_p}$,

$$P[\Omega^1_p, X^{x_0}_t \in A] = E\left[\mathbf{1}_{\Omega^1_p} E\left\{\int_0^t \nu_p(\mathrm{d}s) \int_{\mathbb{R}^d} f_p(s, x)\mathrm{d}x \mathbf{1}_{\{X^x_{t-s} \in A\}}\right\}\right] = 0. \tag{3.41}$$

The last inequality follows from Proposition 3.7, applied with $f_0(x) = f_p(s, x)$ for each s fixed.

Step 4. The conclusion readily follows, copying line by line Step 4 of the proof of Theorem 2.2. □

3.3. Application to some fragmentation equations

We would like to end this paper with an example of application of Corollary 3.3. We will show a regularization property for a class of fragmentation equations. We refer the reader to [6] for details concerning such equations. We describe as a *fragmentation kernel* any non-negative symmetric function $F(x, y) = F(y, x)$ on $(0, \infty) \times (0, \infty)$. A function $c(t, x) : [0, \infty) \times (0, \infty) \mapsto [0, \infty)$, representing the *concentration* of particles with size x at time t is said to solve the fragmentation equation if for all $t \geq 0$, for all $x \in (0, \infty)$,

$$\partial_t c(t, x) = \int_x^\infty F(x, y - x)c(t, y)\mathrm{d}y - \frac{1}{2}c(t, x) \int_0^x F(y, x - y)\mathrm{d}y. \tag{3.42}$$

We will assume in the remainder the following assumptions on the fragmentation kernel (see Remark 3.3 of [6]).

Assumption (K)**.** $F(x, y) = \alpha(x+y)\beta(x/(x+y))$ for some C^1 functions $\alpha : (0, \infty) \mapsto [0, \infty)$ and $\beta : (0, 1) \mapsto [0, \infty)$, β being symmetric at $1/2$.

The conservation of mass $\int_0^\infty xc(t,x)\mathrm{d}x = \int_0^\infty xc(0,x)\mathrm{d}x = 1$ being expected to hold, we may rewrite (3.42) in terms of the probability measures $Q_t(\mathrm{d}x) = xc(t,x)\mathrm{d}x$ (see Definition 2.1 in [6]). It is shown in [6] (see Remark 2.4, Theorem 3.2, Remark 3.3, Proposition 3.8, and Remark 3.10) that the following result holds.

Proposition 3.9. *Assume* (K). *Consider a probability measure* Q_0 *on* $(0,\infty)$, *satisfying* $\langle Q_0, x^p\rangle < \infty$ *for some* $p \geq 1$. *Assume that* $\int_0^1 z(1-z)\beta(z)\mathrm{d}z < \infty$, *and that* $\lim_{x\to 0} x^2\alpha(x) = 0$, *while* $x^2\alpha(x) \leq C(1+x^p)$ *for some constant* C. *Then there exists an* $\mathbb{R}$*-valued Markov process* $\{X_t\}_{t\geq 0}$ *enjoying the following properties:*

(i) *X is a.s. càdlàg, non-increasing, and takes its values in* $[0,\infty)$;
(ii) *the law of* X_0 *is given by* Q_0, *while its generator is given, for any* $\phi \in C_b^1([0,\infty))$, *for any* $y \in (0,\infty)$, *by*

$$L^F(y) = y\alpha(y)\int_0^1 [\phi(y - zy) - \phi(y)]\,(1-z)\beta(z)\mathrm{d}z; \tag{3.43}$$

(iii) *if* $x^2\alpha(x) \leq C(x + x^p)$ *for some constant* C, *then* X *a.s. never reaches* 0, *that is* $P[X_t = 0] = 0$ *for all* $t \geq 0$;
(iv) *if* $x^2\alpha(x) \geq \epsilon x^\delta$ *for some* $\delta \in (0,1)$, *for some* $\epsilon > 0$, *then* $P[X_t = 0] > 0$ *for each* $t > 0$;
(v) *setting* $Q_t = \mathcal{L}(X_t)$ *for each* $t > 0$, *the family* $\{x^{-1}Q_t(\mathrm{d}x)\}_{t\geq 0}$ *solves* (3.42) *in a weak sense.*

We will prove here the following regularization result, which improves [6] Proposition 3.12.

Proposition 3.10. *Additionally to the hypotheses of Proposition 3.9, suppose that for all* $x > 0$, $\alpha(x) > 0$, *and that* $\int_0^1 \beta(z)\mathrm{d}z = \infty$.

1. *Then the law of* X_t *has a density with respect to* $\mathrm{d}x + \delta_0(\mathrm{d}x)$ *as soon as* $t > 0$. *Here* $\mathrm{d}x$ *stands for the Lebesgue measure on* $\mathbb{R}$ *and* $\delta_0(\mathrm{d}x)$ *is the Dirac measure at* 0.
2. *In the case where* $x^2\alpha(x) \leq C(x + x^p)$ *for some constant* C, *this implies that the law of* X_t *has a density with respect to* $\mathrm{d}x$ *as soon as* $t > 0$. *Hence the measure weak solution* $\{x^{-1}Q_t(\mathrm{d}x)\}_{t\geq 0}$ *to* (3.42) *becomes a function weak solution (possibly starting from a measure initial condition).*

Proof. First note that point 2 follows immediately from point 1 and Proposition 3.9(iii). On the other hand, it clearly suffices to prove 1 when $Q_0 = \delta_{x_0}$, for some arbitrary $x_0 > 0$, by linearity.

The Markov process X taking its values in $[0,\infty)$, we just have to check that for each $\epsilon > 0$, for each Lebesgue-null subset $A \subset (\epsilon,\infty)$, for each $t > 0$, $P[X_t \in A] = 0$. Thus let such a couple ϵ, A be fixed.

We unfortunately cannot apply Corollary 3.3 directly, since the map $\gamma(x) = x\alpha(x)$ may explode or vanish when x tends to 0, while $h(x,z) = -xz$ is degenerate when $x = 0$. We thus consider a C_b^1 strictly positive function $\gamma_\epsilon : \mathbb{R} \mapsto (0,\infty)$, and such that $\gamma_\epsilon(y) = \gamma(y)$ for all $y \in [\epsilon, x_0]$ (this is possible since γ is strictly positive and of class C^1 on $(0,\infty)$). Consider also a C_b^1 function $f_\epsilon : \mathbb{R} \mapsto (\epsilon/2,\infty)$, such that $f_\epsilon(y) = y$ for all $y \in [\epsilon, x_0]$. Finally, set $h_\epsilon(y,z) = -f_\epsilon(y)z$. Then there exists a unique Markov process X^ϵ starting from x, non-increasing, with generator

$$L_\epsilon^F(y) = \gamma_\epsilon(y)\int_0^1 [\phi(y + h_\epsilon(y,z)) - \phi(y)]\,(1-z)\beta(z)\mathrm{d}z. \tag{3.44}$$

Noting that $\int_0^1 (1-z)\beta(z)\mathrm{d}z = \infty$ (because β is symmetric at $1/2$ and since $\int_0^1 \beta(z)\mathrm{d}z = \infty$ by assumption), one may easily check that $(A1)$ and $(A3)(y)$ (for any $y \in \mathbb{R}$) holds for X^ϵ. Corollary 3.3 thus ensures that $P[X_t^\epsilon \in A] = 0$ for any $t > 0$.

Finally, X and X^ϵ being almost surely non-increasing, both starting from x_0, and having the same generator for $y \in [\epsilon, x_0]$, clearly coincide while one of them is greater than ϵ (in distribution). Since $A \subset (\epsilon, \infty)$, we deduce that $P[X_t \in A] = P[X_t^\epsilon \in A]$ for any $t > 0$. This concludes the proof. $\square$

References

[1] K. Bichteler, J.B. Gravereaux, J. Jacod, Malliavin calculus for processes with jumps, in: Stochastic Monographs, Number 2, Gordon and Breach, 1987.

[2] K. Bichteler, J. Jacod, Calcul de Malliavin pour les diffusions avec sauts, existence d'une densité dans le cas unidimensionel, in: Séminaire de Probabilités XVII, in: L.N.M., vol. 986, Springer, 1983, pp. 132–157.

[3] J.M. Bismut, Calcul des variations stochastiques et processus de sauts, Z.W. 63 (1983) 147–235.

[4] L. Denis, A criterion of density for solutions of Poisson-driven SDEs, Probab. Theory Related Fields 118 (3) (2000) 406–426.

[5] N. Fournier, Jumping SDEs: absolute continuity using monotonicity, Stochastic Process. Appl. 98 (2) (2002) 317–330.

[6] N. Fournier, J.S. Giet, On small particles in coagulation–fragmentation equations, J. Statist. Phys. 111 (5–6) (2003) 1299–1329.

[7] C. Graham, S. Méléard, Existence and regularity of a solution to a Kac equation without cutoff using Malliavin Calculus, Comm. Math. Phys. 205 (1999) 551–569.

[8] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion processes, North Holland, 1979.

[9] J. Jacod, Equations différentielles linéaires, la méthode de variation des constantes, in: Séminaire de Probabilités XVI, in: L.N.M., vol. 920, Springer Verlag, 1982, pp. 442–448.

[10] D. Nualart, The Malliavin Calculus and Related Topics, Springer Verlag, New York, 1995.

[11] J. Picard, On the existence of smooth densities for jump processes, Probab. Theory Related Fields 105 (4) (1996) 481–511.

[12] P. Protter, Stochastic Integration and Differential Equations, Springer Verlag, Berlin, 1990.